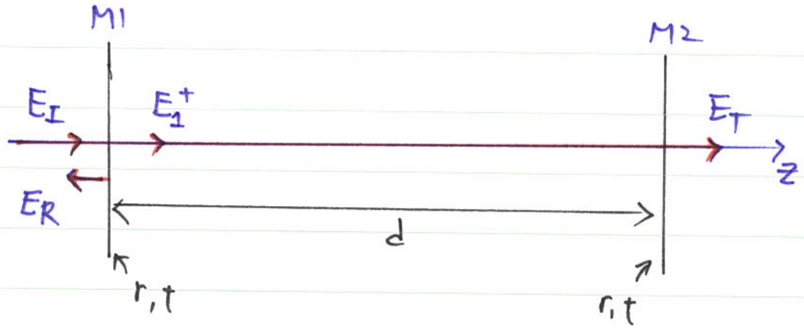
PAGE
DATE 4/11/2016

8-5 Fabry-Perot transmission : the Airy function
(See also 7-9)

· Assume lossless ~~parallel~~ parallel mirror set separated by d.
$(r^2+t^2=1)$



· round trip time τ :

$$\tau = \frac{2d}{v} = \frac{2dn}{c}, \quad v = \frac{c}{n}$$

· How to express E_T (transmitted light) ?

$$E(z_0+\Delta z,\ t_0+\Delta t) = P_F(\Delta z, \Delta t)\, E(z_0, t_0),$$

P_F : propagation factor

$$\rightarrow P_F(\Delta z, \Delta t) = \frac{E(z+\Delta z,\ t_0+\Delta t)}{E(z_0, t_0)} = \frac{E_0 e^{i[\omega(t_0+\Delta t) - k(z_0+\Delta z)]}}{E_0 e^{i(\omega t_0 - k z_0)}}$$

$$= e^{i(\omega\Delta t - k\Delta z)} \quad — ☆$$

· Let $E_I = E_{0I}\, e^{i\omega t}$; $E_1^+ = E_{01}^+(t)\, e^{i\omega t}$ — ☆

· However,
E_1^+ has two components ; transmitted one at $t=t$
returned one (round trip) at $t = t+\tau$

A. For E_1^+;

$$\rightarrow E_1^+(t+\tau) = tE_I(t+\tau) + r^2 P_F(\Delta z = 2d, \Delta t = \tau)\, E_1^+(t) \quad ☆$$

(i) $\therefore E_{01}^{+}(t+\tau)e^{i\omega(t+\tau)} = \underbrace{tE_{0I}e^{i\omega(t+\tau)}}_{\text{just transmitted}} + \underbrace{r^2 E_{01}^{+}(t)e^{i\omega t}e^{i(\omega\tau - 2kd)}}_{\text{round-tripped}}$

· For a steady state (inside the cavity),

$$\rightarrow E_{01}^{+}(t+\tau) = E_{01}^{+}(t) \equiv E_{01}^{+}$$

$$\therefore (i) \rightarrow E_{01}^{+}\cancel{e^{i\omega(t+\tau)}} = tE_{0I}\cancel{e^{i\omega(t+\tau)}} + r^2 E_{01}^{+}\cancel{e^{i\omega t}}\,e^{i(\cancel{\omega\tau} - 2kd)}$$

$$\rightarrow E_{01}^{+}(1 - r^2 e^{-i2kd}) = tE_{0I}$$

$$\therefore E_{01}^{+} = \left(\frac{t}{1 - r^2 e^{-i\delta}}\right)E_{0I},$$

$\delta = 2kd$: round trip phase shift.

B, For E_T :

$$\cdot\ E_T(t + \tfrac{\tau}{2}) = E_{0T}e^{i\omega(\cancel{t} + \tau/2)} = tP_F(\Delta z = d, \Delta t = \tau/2)\,E_{01}^{+}(t)$$

$$= tE_{01}^{+}\cancel{e^{i\omega t}}\,e^{i(\cancel{\omega\tau/2} - \delta/2)}$$

$$= t\left(\frac{t}{1 - r^2 e^{-i\delta}}\right)E_{0I}\,e^{-i\delta/2}$$

-1

$$\therefore E_{0T} = \frac{t^2 e^{-i\delta/2}}{1 - r^2 e^{-i\delta}}\,E_{0I}$$

· Transmission T :

$$T = \frac{I_T}{I_I} = \frac{E_{0T}E_{0T}^{*}}{E_{0I}E_{0I}^{*}}$$

$$\therefore T = \frac{t^4 e^{-i\delta/2} e^{i\delta/2}}{(1-r^2e^{-i\delta})(1-r^2e^{i\delta})}$$

$$= \frac{t^4}{1+r^2-2r^2\cos\delta} = \frac{(1-r^2)^2}{1+r^4-2r^2\cos\delta}$$

Using $\cos\delta = 1-2\sin^2(\frac{\delta}{2})$,

$$1+r^4-2r^2\cos\delta = 1+r^4-2r^2(1-2\sin^2(\tfrac{\delta}{2}))$$

$$= \underbrace{1+r^4-2r^2}\, + \underbrace{4r^2\sin^2(\tfrac{\delta}{2})}$$

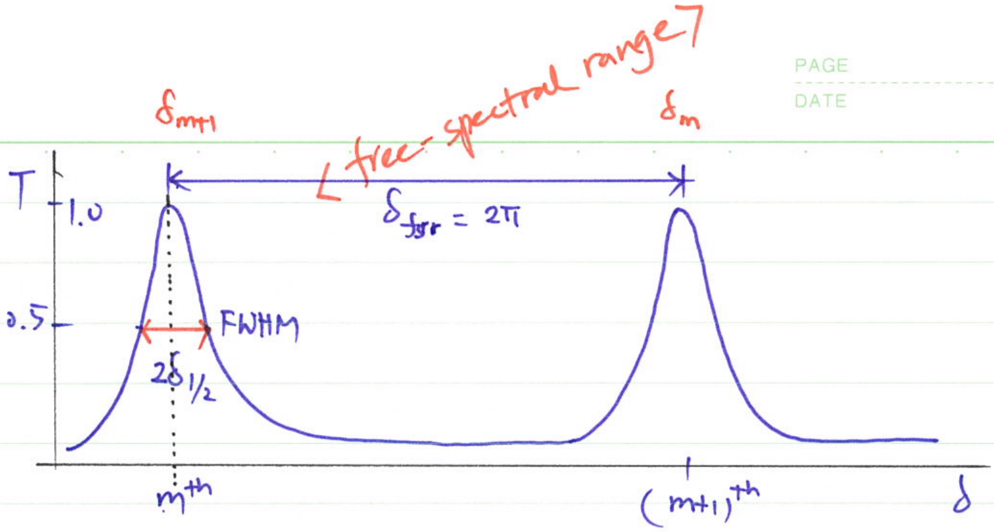
$$\therefore T = \frac{1}{1+\frac{4r^2}{(1-r^2)^2}\sin^2(\frac{\delta}{2})}$$

☆ $F = \dfrac{4r^2}{(1-r^2)^2}$: coefficient of Finesse

$$\therefore T = \frac{1}{1+F\sin^2(\frac{\delta}{2})}$$

- Finesse $\mathcal{F}$, : the ratio of the transmitted peak-to-peak separation to FWHM of the peak.

$$\mathcal{F} = \frac{\pi\sqrt{F}}{2} = \frac{\pi r}{1-r^2}$$



Pf) $I_T = \dfrac{1}{1 + F\sin^2(\frac{\delta}{2})}$ $\rightarrow T_{max} \Rightarrow \sin^2(\frac{\delta}{2}) = 0$ $\therefore \frac{\delta}{2} = m\pi$

$\rightarrow \delta_m = 2m\pi$

$\therefore \delta_{m+1} - \delta_m = 2\pi$

· Half maximum : $\frac{1}{2} I_T$

$\rightarrow F\sin^2(\frac{\delta_{1/2}}{2}) = 1$: ($\delta \rightarrow \delta_{1/2}$ to denote half maximum)

$\rightarrow \sin^2 \frac{\delta_{1/2}}{2} = \frac{1}{F} = \frac{(1-r^2)^2}{4r^2}$

$\rightarrow \sin(\frac{\delta_{1/2}}{2}) = \frac{1-r^2}{2r} \sim \frac{\delta_{1/2}}{2}$ (if $\delta_{1/2} \ll 1$)

$\therefore$ Full width : $2 \cdot \delta_{1/2}$

$2\delta_{1/2} = \frac{1-r^2}{r}$ $\left(\mathcal{F} = \frac{\pi r}{1-r^2} \right)$

$\boxed{\frac{(1+r)(1-r)}{r}}$

(i) $r \sim 0$, $2\delta_{1/2} \sim \frac{1-r}{r} = \frac{1}{r} - 1$

(ii) $r \sim 1$, $2\delta_{1/2} \sim 2(1-r)$

filter.

$\therefore \delta_{1/2} = \pi / \mathcal{F}$

$\frac{\delta_{fsr}}{FWHM} = \frac{2\pi}{2\delta_{1/2}} = \mathcal{F}$

PAGE
DATE

8-6 Scanning Fabry-Perot interferometer : Metrology

· Recall $\delta = 2kd$.

For the max transmitted peaks,

$$\delta = 2m\pi \quad \left(= 2kd_m = 2\frac{2\pi}{\lambda}\cdot d_m\right), \quad m = 0, \pm1, \pm2, \cdots$$

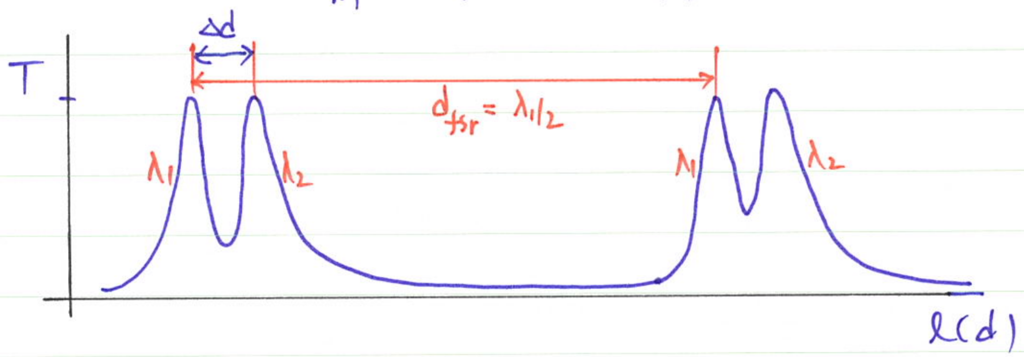
$$\therefore\ d_m = m\lambda/2$$

$$\rightarrow d_{fsr} = d_{m+1} - d_m = \frac{\lambda}{2} \qquad (\cancel{ph.}\ \text{distance})$$

· Remember $\delta_{fsr} = 2\pi$ (phase)

· For two different wavelengths λ_1 & λ_2 incoming to the Fabry-Perot, where $d = 5$ cm, and $\lambda_1 - \lambda_2 \ll \lambda$ ($\lambda \sim \lambda_1, \lambda_2 = 500$ nm),

$$\rightarrow \lambda_1 = \frac{2d_1}{m} \ ; \ \lambda_2 = \frac{2d_2}{m} \qquad (\text{for the same mode})$$



$$\rightarrow \lambda_2 - \lambda_1 = \Delta\lambda = \frac{2}{m}(d_2 - d_1) = \frac{2}{\left(\frac{2d_1}{\lambda_1}\right)}\Delta d$$

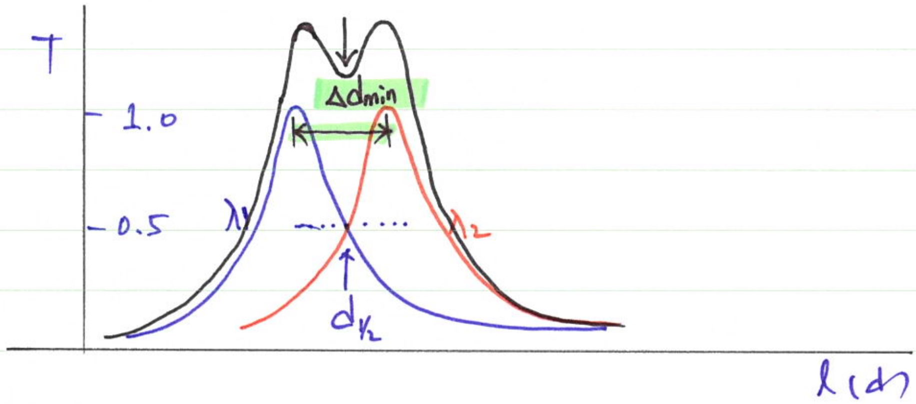
$$\therefore\ \frac{\Delta\lambda}{\lambda_1} = \frac{\Delta d}{d_1}$$

indirect wavelength measurement!

<Resolving Power>

- $\Delta\lambda_{min}$ is limited by $\delta_{1/2}$.

Q. · what is minimum Δd ?



→ Resolution criterion, $\Delta d \geq 2\Delta d_{1/2}$: $(d_{1/2} = \frac{1}{2} d)$

$$= \Delta d_{min}$$

Q. What is minimum $\Delta\lambda$?

$$\rightarrow \quad \mathcal{F} = \frac{d_{fsr}}{2\delta_{1/2}} = \frac{k\, d_{fsr}}{2k\Delta d_{1/2}} = \frac{d_{fsr}}{2\Delta d_{1/2}}$$

$$\therefore \quad 2\Delta d_{1/2} = \frac{d_{fsr}}{\mathcal{F}} = \frac{\lambda}{2\mathcal{F}} \quad (\equiv \Delta d_{min})$$

$$\rightarrow \quad \frac{\Delta\lambda_{min}}{\lambda} = \frac{\Delta d_{min}}{d} \qquad \left(\text{from } \frac{\Delta\lambda}{\lambda} = \frac{\Delta d}{d}\right)$$

$$= \frac{2\Delta d_{1/2}}{d} = \frac{\lambda}{2d\mathcal{F}}$$

Resolving Power, $R \equiv \dfrac{\lambda}{\Delta\lambda_{min}} = \dfrac{2d\mathcal{F}}{\lambda} = m\mathcal{F} \quad \left(m = \dfrac{2d}{\lambda}\right)$

< Interpretation of $\mathcal{R}$. >

→ To get a big $\mathcal{R}$, both m & $\mathcal{F}$ must be large.

However, to maximize m,

the mirror separation d in a Fabry-Perot must be as large as possible.

ex) $r = 0.95$, $d = 1\,cm$, $\lambda = 500\,nm$

Determin mode number, $\mathcal{F}$, and $(\Delta\lambda)_{min}$, $\mathcal{R}$

→ (i) $m = \dfrac{2d}{\lambda} = \dfrac{2 \cdot 10^{-2}}{5 \cdot 10^{-7}} = 40{,}000$

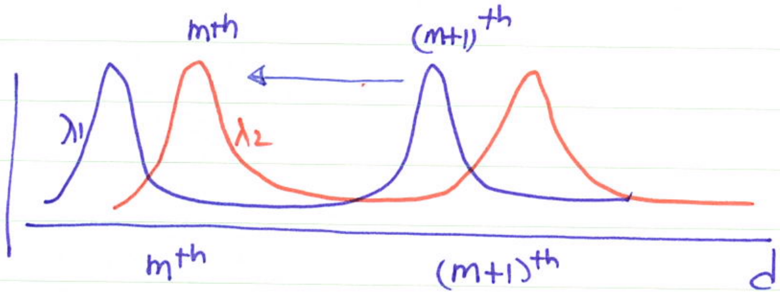
(ii) $\mathcal{F} = \dfrac{\pi r}{1 - r^2} = \dfrac{\pi (0.95)}{1 - 0.95^2} = 31$

(iii) $(\Delta\lambda)_{min} = \dfrac{\lambda}{m\mathcal{F}} = \dfrac{5 \cdot 10^{-7}}{(40{,}000)(31)} = 4 \times 10^{-4}\ (nm)$

(iv) $\mathcal{R} = \dfrac{\lambda}{(\Delta\lambda)_{min}} = \dfrac{500}{4 \times 10^{-4}} = 1.2 \times 10^{6}$

PAGE
DATE

< Maximum $\Delta\lambda$ >



- For max $\Delta\lambda$: $\lambda_1(m+1) = \lambda_2(m)$

(i) for the same mode number

$$\rightarrow \Delta d = m\left(\frac{\lambda_2}{2}\right) - m\left(\frac{\lambda_1}{2}\right) = m\left(\frac{\Delta\lambda}{2}\right)$$

$$d_{fsr} = (m+1)\left(\frac{\lambda_1}{2}\right) - m\left(\frac{\lambda_1}{2}\right) = \frac{\lambda_1}{2}$$

(ii) for the overlap btwn $(m+1)\lambda_1$ & $m\lambda_2$,

$$\rightarrow \Delta d = d_{fsr}$$

$\rightarrow$ overlap occurs if

$$m\left(\frac{\Delta\lambda}{2}\right) = \frac{\lambda_1}{2}$$

$$\therefore \Delta\lambda_{max} = \frac{\lambda_1}{m} \sim \frac{\lambda}{m}$$ (free spectral range of the FP)

$\rightarrow$ To get large $\Delta\lambda$, m must be small

$\rightarrow$ small d is needed!

- $$\frac{\Delta\lambda_{max}}{\Delta\lambda_{min}} = \frac{\left(\frac{\lambda}{m}\right)}{\left(\frac{\lambda}{m\mathcal{F}}\right)} = \mathcal{F}$$

8-7 Variable-input-freq FP interferometer.

· For a fixed $\underbrace{\text{length}}_{\text{cavity}}$ d,

→ round-trip phase shift δ,

$$\delta = 2kd = 4\pi d\left(\frac{\nu}{c}\right) \quad , (\lambda\nu = c)$$

$$\rightarrow \delta_m = 4\pi d\left(\frac{\nu_m}{c}\right), \quad m = 0, \pm 1, \pm 2, \cdots$$

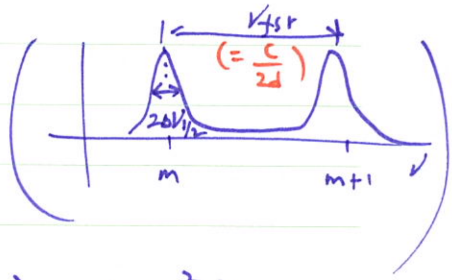
(i) The transmitted peaks occur at

$$\delta_m = 2m\pi \rightarrow \nu_m = \frac{mc}{2d}$$

$$\therefore \nu_{fsr} = \nu_{m+1} - \nu_m = \frac{c}{2d} \quad \text{(free spectral range)}$$

(ii) $2\Delta\nu_{1/2}$

$$\mathcal{F} = \frac{\delta_{frs}}{2\delta_{1/2}} = \frac{\nu_{fsr}}{2\Delta\nu_{1/2}}$$

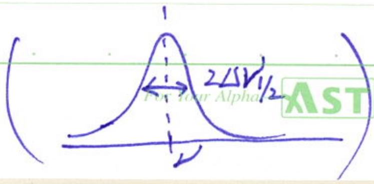


$$\therefore 2\Delta\nu_{1/2} = \frac{\nu_{fsr}}{\mathcal{F}} = \left(\frac{c}{2d}\right)\left(\frac{1-r^2}{\pi r}\right)$$

→ To be used to freq change calibration!

(iii) Quality factor Q,

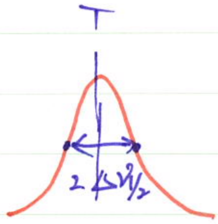
$$Q \equiv \frac{\nu}{2\Delta\nu_{1/2}} = \mathcal{F}\frac{\nu}{\nu_{fsr}}$$



8-8 Lasers & FP cavity

(i) Cavity decay rate, Γ

= cavity loss rate, $2\Delta\nu_{1/2}$



→ For lasing,

this loss must be compensated by the gain medium.

· $E_{01}^{+}(t+\tau) = r^2 E_{01}^{+}(t)$: round trip overlapping

$$E_{01}^{+}(t+\tau) \sim E_{01}^{+}(t) + \tau \frac{d}{dt} E_{01}^{+}(t)$$

$$\rightarrow \frac{d}{dt} E_{01}^{+}(t) = -\left(\frac{1}{\tau}\right)(1-r^2) E_{01}^{+}(t)$$

$$\int \frac{\frac{d}{dt}\left(E_{01}^{+}(t)\right)}{E_{01}(t)} dt = -\int \frac{(1-r^2)}{\tau} dt$$

$$\ln E_{01}^{+}(t) = -\frac{(1-r^2)}{\tau} t + c$$

$$\therefore E_{01}^{+}(t) = E_{01}^{+}(t_0)\, e^{-\frac{1}{\tau}(1-r^2)(t-t_0)}$$

$$\rightarrow I^{+}(t) = E_{01}^{+}(t)\, E_{01}^{+*}(t) = I_{01}^{+}(t_0)\, e^{-\frac{2}{\tau}(1-r^2)(t-t_0)}$$

$$\equiv I_{01}^{+}(t_0)\, e^{-\Gamma(t-t_0)}$$

$$\therefore \Gamma = \frac{2}{\tau}(1-r^2)$$

$$\rightarrow \Gamma \cdot \tau = 2(1-r^2) = 2t^2$$

< Interpretation >

For a highly reflective mirrors,

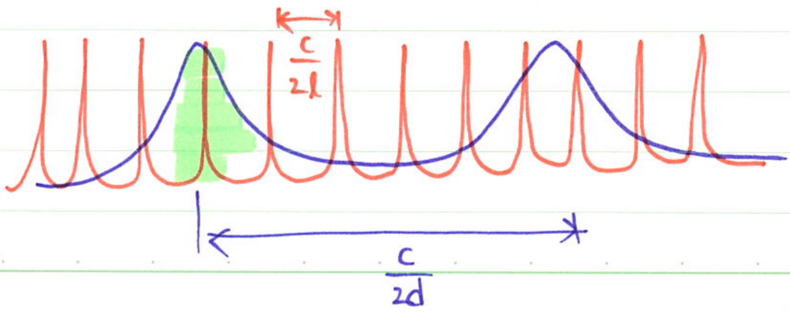
$$\Gamma = \frac{2}{\tau}(1-r^2) = 2\pi r\left(\frac{c}{2d}\,\underbrace{\frac{1-r^2}{\pi r}}_{=2\Delta\nu_{1/2}}\right), \quad \tau = \frac{2d}{c}$$

$$= 2\pi\,(2\Delta\nu_{1/2})$$

$$\rightarrow Q = \frac{2\pi\nu}{\Gamma} = \frac{\omega}{\Gamma} \qquad \left(Q = \frac{\nu}{2\Delta\nu_{1/2}}\right)$$

ex) laser mode suppression

$$\rightarrow \nu_{fsr} = \frac{c}{2l} \ ; \ \frac{c}{2d}$$



ex) Ar-ion laser

- Gain bandwidth : 6 GHz
- Cavity length : 1 m

Q1 Estimate # of longitudinal cavity mode.

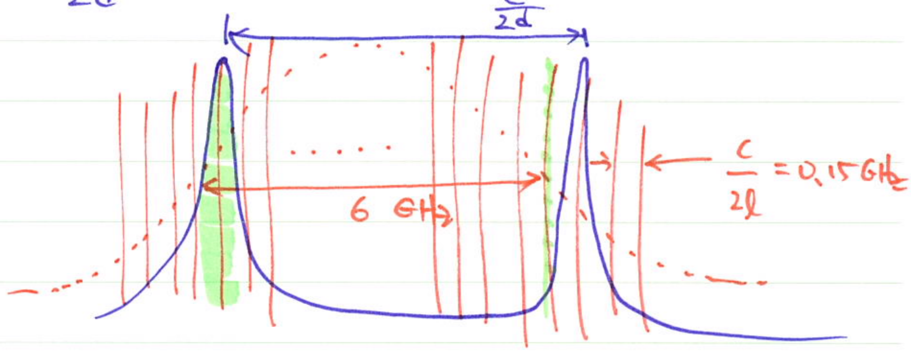
Q2. Find out d of an etalon for a single mode laser output.

→ A1. $\nu_{fsr}^{l} = \frac{c}{2l} = \frac{3\cdot 10^8}{2} = 0.15$ GHz

$\therefore$ # of lasing modes : $\frac{6 \text{ GHz}}{0.15 \text{ GHz}} = 40$

A2. for a single mode operation,

$\nu_{fsr}^{e} = \frac{c}{2d} > 6$ GHz



$\therefore d < \frac{c}{2(6\cdot 10^9)} = 2.5$ cm

* Midterm Exam : Apr. 18, 2016