ScienceDirect

Determinants

A determinant can be expanded in an elementary way in terms of any of its rows or columns:(18)|A|=∑j=1nAijaiji=fixedis the expansion according to the ith row (we sum over all columns): (19)|A|=∑i=1nAijaijj=fixedis the expansion according to the jth column (we sum over all rows).

From: Elementary Molecular Quantum Mechanics (Second Edition), 2013

Related terms:

Polynomial, Tensor, Eigenvalues, Det, Eigenvector, Square Matrix, wavefunction ψ

Determinants

William Ford, in Numerical Linear Algebra with Applications, 2015

The determinant is defined for any $n \times n$ matrix and produces a scalar value. You have probably dealt with determinants before, possibly while using Cramer's rule. The determinant has many theoretical uses in linear algebra. Among these is the definition of eigenvalues and eigenvectors, as we will see in Chapter 5. In Section 4.1, we will develop a formula for the inverse of a matrix that involves a determinant. A matrix is invertible if its determinant is not zero. In vector calculus, the Jacobian matrix is the matrix of all first-order partial derivatives of a multivariate function. The determinant of the Jacobian matrix, called the Jacobian, is used in multivariable calculus.

Read full chapter

URL: https://www.sciencedirect.com/science/article/pii/B9780123944351000041

Determinants

Vincent Pavan, in Exterior Algebras, 2017

5.2.4 Expansion by blocks

Theorem 5.4 (Calculation of determinants by blocks)

Consider a family $\mathbf{x}_1, \cdots, \mathbf{x}_n$ of n vectors of $\boldsymbol{E}$ equipped with a basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$. We suppose that there exists an integer $p \in [1, n-1]$ such that:

$$\forall k \leq p, \quad \mathbf{x}_k = \sum_{j=1}^{j=p} x_{j,k}\mathbf{e}_j. \qquad [5.44]$$

We then consider the strictly ordered $n-p$-index $\mathbf{i} \coloneqq (p+1, \cdots, n) \in \mathscr{I}_o(n-p)$ and the family of reduced vectors $\tilde{\mathbf{x}}_{p+1}(\mathbf{i}), \cdots, \tilde{\mathbf{x}}_n(\mathbf{i})$, we then have the equality:

$$\begin{aligned}&\det_{\mathbf{e}_1,\cdots,\mathbf{e}_n}(\mathbf{x}_1,\cdots,\mathbf{x}_n)\\ &\quad = \det_{\mathbf{e}_1,\cdots,\mathbf{e}_p}(\mathbf{x}_1,\cdots,\mathbf{x}_p)\det_{\mathbf{e}_{p+1},\cdots,\mathbf{e}_n}(\tilde{\mathbf{x}}_{p+1}(\mathbf{i}),\cdots,\tilde{\mathbf{x}}_n(\mathbf{i})).\end{aligned} \qquad [5.45]$$

Proof.– The proof is quite immediate within the formalism of exterior algebras. Indeed, due to the particular form of vectors $\mathbf{x}_1, \cdots, \mathbf{x}_p$, we calculate at first:

$$\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p = \det_{\mathbf{e}_1,\cdots,\mathbf{e}_p}(\mathbf{x}_1,\cdots,\mathbf{x}_p)\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p. \qquad [5.46]$$

Let us now develop the exterior product of the other vectors:

$$\mathbf{x}_{p+1} \wedge \cdots \wedge \mathbf{x}_n = (\mathbf{x}_{p+1} - \tilde{\mathbf{x}}_{p+1}(\mathbf{i}) + \tilde{\mathbf{x}}_{p+1}(\mathbf{i})) \wedge \cdots \wedge (\mathbf{x}_n - \tilde{\mathbf{x}}_n(\mathbf{i}) + \tilde{\mathbf{x}}_n(\mathbf{i})), \qquad [5.47]$$

it is clear, in virtue of the very construction, that each vector of the form $\mathbf{x}_j - \tilde{\mathbf{x}}_j(\mathbf{i}), j \geq p+1$ admits only the components over the vectors $\mathbf{e}_1, \cdots, \mathbf{e}_p$. Therefore, by developing the term above, thanks to the multi-linearity of symbols $\wedge$, there will be only one symbol without terms over $\mathbf{e}_1, \cdots, \mathbf{e}_p$, that is:

$$\tilde{\mathbf{x}}_{p+1}(\mathbf{i}) \wedge \cdots \wedge \tilde{\mathbf{x}}_n(\mathbf{i}) = \det_{\mathbf{e}_{p+1},\cdots,\mathbf{e}_n}(\tilde{\mathbf{x}}_{p+1}(\mathbf{i}),\cdots,\tilde{\mathbf{x}}_n(\mathbf{i}))\mathbf{e}_{p+1} \wedge \cdots \wedge \mathbf{e}_n. \quad [5.48]$$

Therefore, we immediately obtain:

$$\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p \wedge (\mathbf{x}_{p+1} \wedge \cdots \wedge \mathbf{x}_n) = \det_{\mathbf{e}_1,\cdots,\mathbf{e}_p}(\mathbf{x}_1,\cdots,\mathbf{x}_p)\mathbf{e}_1 \wedge \cdots \mathbf{e}_p \wedge (\tilde{\mathbf{x}}_{p+1}(\mathbf{i}) \wedge \cdots \wedge \tilde{\mathbf{x}}_n(\mathbf{i})). \quad [5.49]$$

Hence the announced result, by replacing $\tilde{\mathbf{x}}_{p+1}(\mathbf{i}) \wedge \cdots \wedge \tilde{\mathbf{x}}_n(\mathbf{i})$ with formula [5.48].

Read full chapter

URL: https://www.sciencedirect.com/science/article/pii/B9781785482373500053

Matrix Algebra

S.M. Blinder, in Guide to Essential Math (Second Edition), 2013

9.3 Determinants

Determinants, an important adjunct to matrices, can be introduced as a geometrical construct. Consider the parallelogram shown in Figure 9.1, with one vertex at the origin $(0,0)$ and the other three at (x_1,y_1), (x_2,y_2), and (x_1+x_2,y_1+y_2). Using Pythagoras' theorem, the two sides a,b and the diagonal c have the lengths

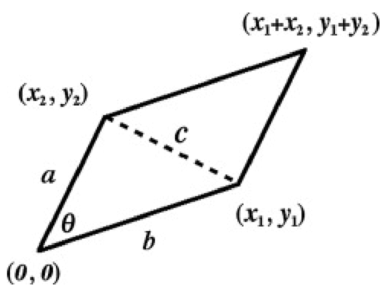


Figure 9.1. Area of parallelepiped equals determinant $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$.

$$a = \sqrt{x_1^2 + y_1^2}, \quad b = \sqrt{x_2^2 + y_2^2}, \quad c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad (9.27)$$

The area of the parallelogram is given by

$$\pm A = ab \sin\theta, \quad (9.28)$$

where θ is the angle between sides a and b. The $\pm$ sign is determined by the relative orientation of (x_1,y_1) and (x_2,y_2). Also, by the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \quad (9.29)$$

Eliminating θ between Eqs. (9.28) and (9.29), we find, after some lengthy algebra, that

$$\pm A = x_1 y_2 - y_1 x_2. \quad (9.30)$$

(If you know about the cross product of vectors, this follows directly from $A = \mathbf{a} \times \mathbf{b} = ab\sin\theta = x_1y_2 - y_1x_2$.) This combination of variables has the form of a *determinant*, written

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - y_1x_2. \quad (9.31)$$

In general for a 2×2 matrix $\mathbb{M}$

$$\det \mathbb{M} = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21}. \quad (9.32)$$

The three-dimensional analog of a parallelogram is a parallelepiped, with all six faces being parallelograms. As shown in Figure 9.2, the parallelepiped is oriented between the origin $(0,0,0)$ and the point $(x,y,z) = (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$, which is the reflection of the origin through the plane containing the points (x_1,y_1,z_1), (x_2,y_2,z_2), and (x_3,y_3,z_3). You can figure out, using some algebra and trigonometry, that the volume is given by

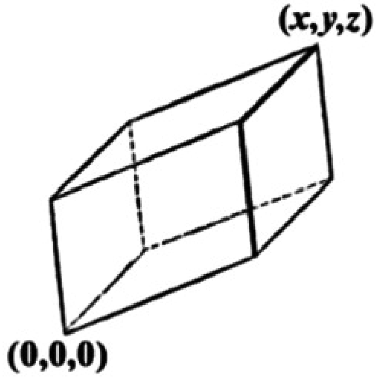


Figure 9.2. Volume of parallelepiped equals determinant $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$.

$$\pm V = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1 y_2 z_3 + y_1 z_2 x_3 + z_1 x_2 y_3 - x_3 y_2 z_1 - y_3 z_2 x_1 - z_3 \tag{9.33}$$

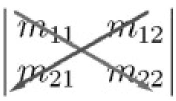
[Using vector analysis, $\pm V = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$, where $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$ are the vectors from the origin to (x_1,y_1,z_1),(x_2,y_2,z_2),(x_3,y_3,z_3), respectively.]

It might be conjectured that an $n \times n$-determinant represents the hypervolume of an n-dimensional hyperparallelepiped.

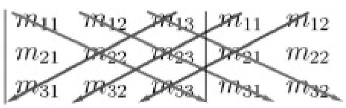
In general, a 3×3 determinant is given by

$$\det \mathbb{M} = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} = m_{11}m_{22}m_{33} + m_{12}m_{23}m_{31} + m_{13}m_{21}m_{32} \\ - m_{13}m_{22}m_{31} - m_{12}m_{21}m_{33} - m_{11}m_{23}m_{32}. \tag{9.34}$$

A 2×2 determinant can be evaluated by summing over products of elements along the two diagonals, northwest-southeast minus northeast-southwest:



Similarly for a 3×3 determinant:



where the first two columns are duplicated on the right. There is no simple graphical method for 4×4 or larger determinants. An $n \times n$ determinant is defined more generally by

$$\det \mathbb{M} = \sum_{p=1}^{n!} (-1)^p \mathscr{P}[m_{1i} m_{2j} m_{3k} \cdots], \tag{9.35}$$

where $\mathscr{P}$ is a *permutation* operator which runs over all $n!$ possible permutations of the indices $i,j,k,\ldots$ The permutation label p is even or odd, depending on the number of binary interchanges of the second indices necessary to obtain $m_{1i}\, m_{2j}\, m_{3k} \cdots$, starting from its order on the main diagonal: $m_{11}\, m_{22}\, m_{33} \cdots$. Many math books show further reductions of determinants involving *minors* and *cofactors*, but this is no longer necessary with readily available computer programs to evaluate determinants. An important property of determinants, which is easy to verify in the 2×2 and 3×3 cases, is that if any two rows or columns of a determinant are interchanged, the value of the determinant is multiplied by -1. As a corollary, if any two rows or two columns are identical, the determinant equals zero.

The determinant of a product of two matrices, in either order, equals the product of their determinants. More generally for a product of three or more matrices, in any cyclic order,

$$\det(\mathbb{ABC}) = \det(\mathbb{BCA}) = \det(\mathbb{CAB}) = \det \mathbb{A} \det \mathbb{B} \det \mathbb{C}. \tag{9.36}$$

Problem 9.3.1

Find the volume of a unit cube coincident with the coordinate axes by evaluating a 3×3 determinant.

Problem 9.3.2

As a more challenging variant, calculate the volume of a rotated unit cube with one vertex standing on the origin.

Read full chapter

URL: https://www.sciencedirect.com/science/article/pii/B9780124071636000096

Linear Algebra

Seifedine Kadry, in Mathematical Formulas for Industrial and Mechanical Engineering, 2014

3.5 Determinants

The determinant of a square matrix A is denoted by the symbol $|A|$ or $\det A$. We can form determinants of $n \times n$ matrices. Such determinants are called $n \times n$ determinants.

Definition

1. If $A = [a_{11}]$ is a 1×1 matrix, then its determinant $|A|$ is equal to the number a_{11} itself.
2. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix, then the determinant is given by

$$\left|A\right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

3. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3×3 matrix, then its determinant is given by

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{3} \end{aligned}$$

Note that the determinant is expanded along the first row. Similarly, the determinant can be expanded along any other row or column carrying (+) or (–) sign according to the place occupied by the element in the following scheme:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Example

1. $|2| = 2$
2. $\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = 2 \times (-5) - 3 \times 4 = -10 - 12 = -22$
3. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \\ 3 & -4 & -6 \end{vmatrix} = 1 \begin{vmatrix} 1 & 3 \\ -4 & -6 \end{vmatrix} - 3 \begin{vmatrix} 2 & 3 \\ 3 & -6 \end{vmatrix} + 5 \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix}$, expanding along the first row

$$= 1(-6 + 12) - 3(-12 - 9) + 5(-8 - 3) = 6 + 63 - 55 = 14$$

Read full chapter

URL: https://www.sciencedirect.com/science/article/pii/B9780124201316000038

Determinants and Matrices

George B. Arfken, ... Frank E. Harris, in Mathematical Methods for Physicists (Seventh Edition), 2013

The determinant is defined and it is shown how it is used in the solution of sets of simultaneous linear equations. Matrices and elementary matrix operations are then introduced, including addition, multiplication by a scalar, and matrix multiplication. Zero and unit matrices are defined, as are the inverse and the determinant of a matrix. Matrix singularity is discussed and related to the vanishing of the determinant of the matrix. The determinant product theorem is stated and proved. The transpose, adjoint, and trace of a matrix are defined, following which the text defines orthogonal, unitary, and Hermitian matrices. The Pauli and Dirac matrices are introduced as illustrations of anticommuting sets of matrices. Functions of matrices are defined.

Keywords: determinant, Cramer's rule, expansion of determinant in minors, matrices, matrix addition, matrix multiplication by scalar, matrix multiplication, transpose matrix, adjoint matrix, Hermitian matrix, orthogonal matrix, unitary matrix, anticommuting matrices, Pauli matrices, Dirac matrices, trace formula.

Read full chapter

URL: https://www.sciencedirect.com/science/article/pii/B9780123846549000025

MATHEMATICAL PRELIMINARIES

ByPál Rózsa, in Applied Dimensional Analysis and Modeling (Second Edition), 2007

Definition 1-1.

The determinant of a matrix **A** *is*

$$|\mathbf{A}| = \sum_{n!}(-1)^q \cdot a_{1j_1} \cdot a_{2j_2} \ldots a_{nj_n}$$

where q is the *number of inversions* in the *permutation* set $\mathbf{j}_1\, \mathbf{j}_2 \ldots \mathbf{j}_n$ *for numbers 1, 2,* ..., n, *which are summed over all* n! *permutations of the first* n *natural numbers.* (For example, the number of inversions in the permutation 35214 is 6, since 3 is in inversion with 2 and 1; 5 with 2, 1, 4; and 2 with 1).

According to this definition, any determinant of order 3 can be calculated in the following way:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32} - a_{12} \cdot a_{21} \cdot a_{33} + a_{12} \cdot (\ldots$$
$$-a_{13} \cdot a_{22} \cdot a_{31}$$

Factoring out a_{11}, a_{12} and a_{13}, we can write for the above determinant

$$|\mathbf{A}| = a_{11} \cdot (a_{22} \cdot a_{33} - a_{23} \cdot a_{32}) - a_{12} \cdot (a_{21} \cdot a_{33} - a_{23} \cdot a_{31}) + a_{13} \cdot (a_{21} \ldots$$

The expressions in parentheses are the *determinants* of the second order, namely

$$a_{22} \cdot a_{33} - a_{23} \cdot a_{32} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}; \; a_{21} \cdot a_{33} - a_{23} \cdot a_{31} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix};$$
$$a_{21} \cdot a_{32} - a_{22} \cdot a_{31} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Therefore determinant |**A**| can be written as

$$|\mathbf{A}| = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (1\text{-}2)$$

Note that the elements of the first row of |**A**| are multiplied by the second-order determinants, which are obtained by *omitting* the first row and the corresponding column of |**A**|, and then affixing a negative sign to the second determinant. Expression (1-2) is called the *expansion* of the determinant by its first row. This technique can be generalized for any determinant of arbitrary order *n*, and, in fact, it is more useful, understandable, and much more practical to use than Definition 1-1. Thus, for the general case we proceed as follows:

First, we define the concept of *cofactors*. To any element a_{ij} of a determinant of order *n* can be assigned a *subdeterminant* of order *n* – 1, by *omitting* the *i*th row and the *j*th column of the determinant. Then we assign the sign $(-1)^{i+j}$ to it (the "chessboard" rule: white squares are positive, black squares are negative) and the result is the *cofactor* denoted by A_{ij}. Note that the cofactor A_{ij} already includes the appropriate sign!

Read full chapter

URL: https://www.sciencedirect.com/science/article/pii/B9780123706201500071

H

Fred E. Szabo PhD, in The Linear Algebra Survival Guide, 2015

Illustration

Determinants can be used to classify critical points of differentiate functions. For example, if $f: \mathbb{R}^2 \to \mathbb{R}$ is a function with continuous second partial derivatives f_{xx}, f_{xy}, f_{yx}, and f_{yy}, then the matrix

MatrixForm [H_f = {{f_{xx}, f_{xy}}, {f_{yx}, f_{yy}}}]

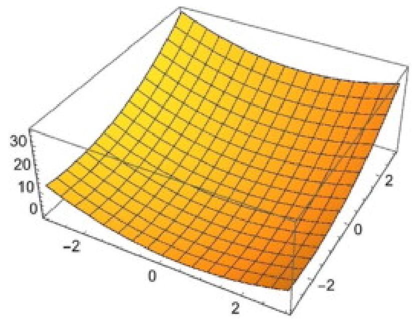
$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

is a Hessian matrix. Its determinant *Det[H_f]* is called the discriminant of f. We know from calculus that if $\boldsymbol{v}$ = {x, y} is a critical point of f, in other words, if $f_x[\boldsymbol{v}] = f_y[\boldsymbol{v}] = 0$ and *Det[H_f]* > 0, then *f*[**v**] is a local minimum of f if $f_{xx}[\boldsymbol{v}] > 0$ and a local maximum if $f_{xx}[v] < 0$. If *Det[H_f]* < 0, then $\boldsymbol{v}$ is a saddle point of f. If *Det[H_f]* = 0, the test fails.

- Hessian matrix

  f [x_, y_] : = x^2 + y^2 - x + 4y + 3;

  Plot3D[f[x, y], {x, – 3, 3}, {y, – 3, 3}]




```
fxx = D[f[x, y] , x, x] ; fxy = D[f[x, y] , x, y] ;
fyx = D[f[x, y] , y, x] ; fyy = D[f[x, y] , y, y] ;
Hf = {{fxx, fxy}, {fyx, fyy}}
{{2, 0}, {0, 2}}
Det[Hf]
4
{Reduce[D[f[x, y], x] == 0, {x, y}], Reduce[D[f[x, y], y]==
0, {x, y}]}
```

$$\left\{x == \frac{1}{2},\ y = -2\right\}$$

Since the determinant of $H_f > 0$, *f* has a local minimum at {x, y} = {1/2, – 2}.

Read full chapter

URL: https://www.sciencedirect.com/science/article/pii/B9780124095205500151

Systems of linear equations, matrices, and determinants

Mary Attenborough, in Mathematics for Electrical Engineering and Computing, 2003

The inverse of a matrix

The inverse of a matrix A is a matrix $\mathbf{A}^{-1}$ such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ (the unit matrix).

Example 13.10

Show that

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

is the inverse of

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}.$$

Solution Multiply:

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(2)+\frac{1}{3}(1) & \frac{1}{3}(1)+\frac{1}{3}(-1) \\ \frac{1}{3}(2)+\left(-\frac{2}{3}\right)(1) & \frac{1}{3}(1)+\left(-\frac{2}{3}\right)(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} (2)\frac{1}{3}+(1)\frac{1}{3} & (2)\frac{1}{3}+(1)\left(-\frac{2}{3}\right) \\ (1)\frac{1}{3}+(-1)\frac{1}{3} & (1)\frac{1}{3}+(-1)\left(-\frac{2}{3}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Not all matrices have inverses and only square matrices can possibly have inverses. A matrix does not have an inverse if its determinant is 0.

The determinant of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

If the determinant of a matrix is 0 then it has no inverse and the matrix is said to be singular. If the determinant is non- zero then the inverse exists. The inverse of the 2 × 2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$\frac{1}{(ad-cb)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

That is, to find the inverse of a 2 × 2 matrix, we swap the diagonal elements, negate the off-diagonal elements, and divide the resulting matrix by the determinant.

Example 13.11

Find the determinants of the following matrices and state if the matrix has an inverse or is singular. Find the inverse in the cases where is exists and check that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

$$\text{(a)} \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}, \text{(b)} \begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix}, \text{(c)} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Solution

$$\text{(a)} \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = (-1)\times 1 - 2\times 3 = -7.$$

As the determinant is not zero the matrix

$$\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$$

has an inverse found by swapping the diagonal elements and negating the off-diagonal elements, then dividing by the determinant. This gives

$$\frac{1}{-7}\begin{pmatrix} 1 & -3 \\ -2 & -1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}.$$

Check that $\mathbf{AA}^{-1} = \mathbf{I}$

$$\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}\frac{1}{7}\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} (-1)(-1)+(3)(2) & (-1)3+(3)(1) \\ (2)(-1)+(1)(2) & (2)(3)+(1)(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

$$\frac{1}{7}\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$$
$$= \frac{1}{7}\begin{pmatrix} (-1)(-1)+(3)(2) & (-1)3+(3)(1) \\ (2)(-1)+(1)(2) & (2)(3)+(1)(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\text{(b)}\begin{vmatrix} 6 & -2 \\ -3 & 1 \end{vmatrix} = 6 \cdot 1 - (-3)(-2) = 0$$

As the determinant is zero the matrix

$$\begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix}$$

has no inverse. It is singular.

$$\text{(c)}\begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$
$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = 1.$$

Therefore, the matrix is invertible. Its inverse is given by swapping the diagonal elements, and negating the off-diagonal elements, and then dividing by the determinant. This gives

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Check that **AA** $^{-1}$ =**I**:

$$\mathbf{AA}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly, **A**$^{-1}$ **A** =**I.**

Read full chapter

URL: https://www.sciencedirect.com/science/article/pii/B9780750658553500395

ELSEVIER

Copyright © 2020 Elsevier B.V. or its licensors or contributors.
ScienceDirect ® is a registered trademark of Elsevier B.V.

RELX™