EC5103 Midterm exam April 17, 2017

Name:

1. (24 points) From the following two waves,
   $E_1 = E_0\cos(k_1 x - \omega_1 t)$; $E_2 = E_0\cos(k_2 x - \omega_2 t)$, where $\omega_1 \approx \omega_2$,
   (i) Derive phase velocity ω_P and group velocity ω_g using superposition in vacuum.
   (ii) Inside a medium whose refractive index n(k) is wavelength dependent, prove that
   $v_g = v_P\left[1 + \frac{\lambda}{n}\left(\frac{dn}{d\lambda}\right)\right]$.
   (iii) In regions of normal dispersion, $\frac{dn}{d\lambda} < 0, v_g < v_P$. Discuss if $v_g > v_P$ is possible?
2. (32 points) In an optical cavity composed of two plain mirrors separated by d, suppose there is the following wave initially: $E_1 = E_0\sin(\omega t + kx)$.
   (i) What is the propagation direction of this wave?
   (ii) When it hits on the mirror it reflects backward. Describe the reflected wave and discuss its propagation direction.
   (iii) When two waves are superposed it forms a standing wave. Describe it in a wave form and discuss its propagation direction.
   (iv) According to electromagnetism, the electric field on the surface of a dielectric medium is zero. From this condition, find out normal modes of the cavity for standing wave.
3. (32 points) In a Fabry-Perot interferometer composed of two separated plain mirrors, whose separation is d,
   (i) Describe the transmitted light intensity I_T in terms of reflection coefficient r and phase difference δ between successive waves.
   (ii) For r=0.9, find out FWHM (Full width at half maximum) of the transmitted light.
   (iii) What is the ratio of FWHM in (ii) with respect to the free spectral range (between successive transmitted light peaks)?
   (iv) The FWHM becomes a resolution criterion. If two different waves are to be resolved, what is the minimum $\Delta\lambda$ in terms of $\delta_{1/2}$ in (ii) and λ?
4. (16 points) A gain medium in put in the optical cavity shown in #3(ii), where the gain bandwidth G is very broad to cover several hundreds of cavity modes.
   (i) Discuss a method to make a single mode laser. Hint: Use d_{fsr} of the cavity
   (ii) If a Fabry-Perot etalon is used, whose etalon separation is s, what is s in terms of d?
   You need to show them with a sketch of each mode.

Sum-Product Identities

(20) $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

(21) $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

(22) $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

(23) $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

Solution

1. $E_1 = E_0 \cos(k_1 x - \omega_1 t)$, $E_2 = E_0 \cos(k_2 x - \omega_2 t)$; $\omega_1 \sim \omega_2$

(i) $E_R = E_1 + E_2 = E_0 [\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)]$

Using $\cos\alpha + \cos\beta \equiv 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$,

$\alpha = k_1 x - \omega_1 t$; $\beta = k_2 x - \omega_2 t$

$\rightarrow \alpha+\beta = (k_1+k_2)x - (\omega_1+\omega_2)t$; $\alpha-\beta = (k_1-k_2)x - (\omega_1-\omega_2)t$

$\sim 2k_p x - 2\omega_p t$ $\quad$ $= \Delta k x - \Delta\omega t$

$\equiv 2k_g x - 2\omega_g t$

then $E_R = 2E_0 \cos(k_p x - \omega_p t)\cos(k_g x - \omega_g t)$

$\rightarrow k_p = \frac{k_1+k_2}{2}$; $\omega_p = \frac{\omega_1+\omega_2}{2}$; $k_g = \frac{k_1-k_2}{2}$; $\omega_g = \frac{\omega_1-\omega_2}{2}$

For ~~...~~ $k_p x - \omega_p t \equiv \phi$, ① $\frac{dx}{dt} = v_p = \frac{\omega_p}{k_p} = \frac{\omega_1+\omega_2}{k_1+k_2} \sim \frac{\omega}{k}$

② $v_g = \frac{\omega_g}{k_g} = \frac{\omega_1-\omega_2}{k_1-k_2} = \frac{\Delta\omega}{\Delta k} \sim \frac{d\omega}{dk}$

(ii) From (i), $\omega = k v_p$

$\therefore v_g = \frac{d\omega}{dk} = \frac{d}{dk}(k v_p) = v_p + k\frac{dv_p}{dk}$

In a dispersive medium, $v_p = \frac{c}{n}$.

$$\therefore v_g = v_p + k\frac{d}{dk}\left(\frac{c}{n}\right) = v_p + kc\left(-\frac{1}{n^2}\right)\frac{dn}{dk}$$
$$= v_p - v_p\left(\frac{k}{n}\right)\frac{dn}{dk}$$
$$= v_p\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

Here, $k = \frac{2\pi}{\lambda}$ $\quad\therefore dk = -\frac{2\pi}{\lambda^2}d\lambda$

$$\therefore v_g = v_p\left[1 + \frac{\lambda}{n}\left(\frac{dn}{d\lambda}\right)\right]$$

(iii) For abnormal regions, $\frac{dn}{d\lambda} > 0$, $v_g > v_p$.

2.

$E_1 = E_0 \sin(\omega t + kx)$

i) $\phi = \omega t + kx$, $\quad d\phi = 0 = \omega\, dt + k\, dx$

$$\therefore \frac{dx}{dt} = -\frac{\omega}{k} < 0$$

$\rightarrow$ Left propagation

ii) Reflected wave(E_2) must propagate to the right direction. with a π-phase shift.

$$E_2 = E_0 \sin(\omega t - kx + \pi) = -E_0 \sin(\omega t - kx)$$

(iii) $E_R = E_1 + E_2 = E_0 \left[\sin(\omega t + kx) - \sin(\omega t - kx) \right]$

Using $\sin\alpha - \sin\beta \equiv 2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$,

$\rightarrow$ $E_R = 2E_0 \cos\omega t \sin kx$: ~~Star~~

$\rightarrow$ Standing wave (No propagation)

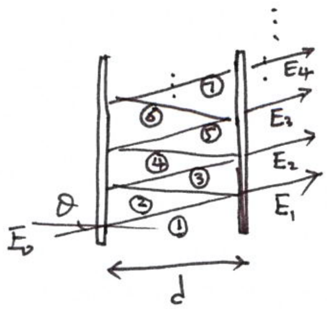
(iv) From (iii), the zero condition is

$$kx = m\pi \longrightarrow x_m = \frac{m\pi}{k} = m\frac{\lambda}{2} \quad (m = 0, 1, 2, \cdots)$$

x_m is called normal modes.

3.

(i)



$$① \; tE_0 e^{i\delta} \rightarrow \quad E_1 = t^2 E_0 e^{i\delta} e^{i\omega t}$$

$$\therefore \cancel{E_n = t^2 r^{2(n-1)} E_0 e^{i(2n-1)\delta}, \; n=1,2,3,\dots}$$

$$\delta = d\cos\theta$$

$$② \; trE_0 e^{2i\delta} \rightarrow$$

$$③ \; tr^2 E_0 e^{3i\delta} \rightarrow \quad E_2 = t^2 r^2 E_0 e^{3i\delta} e^{i\omega t}$$

$$④ \; tr^3 E_0 e^{4i\delta}$$

$$⑤ \; tr^4 E_0 e^{5i\delta} \rightarrow \quad E_3 = t^2 r^4 E_0 e^{5i\delta} e^{i\omega t}$$

$$⑥ \; tr^5 E_0 e^{6i\delta}$$

$$⑦ \; tr^6 E_0 e^{7i\delta} \rightarrow \quad E_4 = t^2 r^6 E_0 e^{7i\delta} e^{i\omega t}$$

$$\underline{\delta = d\cos\theta \, k}$$

$$\vdots \qquad \vdots$$

$$E_n = t^2 r^{2(n-1)} E_0 e^{i(2n-1)\delta} e^{i\omega t}$$

$$\therefore E_T = \sum_{n=1}^{\infty} E_n = t^2 E_0 e^{i\omega t} \sum_{n=1}^{\infty} r^{2(n-1)} e^{i(2n-1)\delta}$$

$$= t^2 E_0 e^{i\delta} e^{i\omega t} \sum_{n=1}^{\infty} r^{2(n-1)} e^{i2(n-1)\delta}$$

Let $x \equiv r^2 e^{2i\delta}$.

$$\rightarrow E_T = t^2 E_0 e^{i\delta} e^{i\omega t} \sum_{n=2}^{\infty} x^{n-2}$$

$$\sum_{n=2}^{\infty} x^{n-2} = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

$$\therefore E_T = t^2 E_0 e^{i\delta} e^{i\omega t} \Big/ (1 - r^2 e^{2i\delta}).$$

$$I_T = E_T E_T^* = \frac{t^4 I_0}{(1 - r^2 e^{2i\delta})(1 - r^2 e^{-2i\delta})} = \frac{t^4 I_0}{1 + r^4 - 2r^2(e^{2i\delta} + e^{-2i\delta})}$$

$$= \frac{t^4 I_0}{1 + r^4 - 2r^2 \cos 2\delta} \; ; \quad t^4 = (1 - r^2)^2$$

For maximum, $\cos 2\delta = 1 \rightarrow 2\delta = 2m\pi = 2d\frac{2\pi}{\lambda}\cos\theta$

$$\rightarrow \underline{2d\cos\theta = m\lambda}.$$

$$\underline{\underline{I_T = I_0}}$$

(ii) $r = 0.9$

$$I_T = \frac{(1-r^2)^2}{1 + r^4 - 2r^2 \cos 2\delta} I_0 .$$

Here $\cos 2\delta = \cos(\delta + \delta) = \cos^2\delta - \sin^2\delta = 1 - 2\sin^2\delta$

$$\therefore I_T = \frac{(1-r^2)^2}{1 + r^4 - 2r^2 + 4r^2\sin^2\delta} I_0 = \frac{I_0}{1 + \frac{4r^2}{(1-r^2)^2}\sin^2\delta}$$

$$= \frac{I_0}{1 + F\sin^2\delta} \quad ; \quad F \equiv \frac{4r^2}{(1-r^2)^2}$$

For the FWHM, $I_T \rightarrow \frac{1}{2} I_T$, $F\sin^2\delta = 1$

With $r = 0.9$, $F = \frac{4(0.9)^2}{(1-0.9^2)^2} = \frac{3.24}{0.0361} = 89.75$

$$\therefore \sin^2\delta_{1/2} = \frac{1}{F} = \frac{1}{89.75} = 0.011$$

$$\rightarrow \sin\delta_{1/2} = 0.106 \ll 1$$

$$\therefore \sin\delta_{1/2} \sim \delta_{1/2} = d\cos\theta K \sim \frac{2\pi d}{\lambda} = 0.106$$

$$\delta_{1/2} = \frac{1}{\sqrt{F}}$$

$$\therefore \text{FWHM} = 2\delta_{1/2} = \frac{2}{\sqrt{F}} = 0.21$$

(iii) $$\frac{\text{FWHM}}{\delta_{fsr}} = \frac{0.21}{2\pi} \sim \underline{\underline{0.033}}$$

(iv) From maximum transmission condition in (i) for normal incidence,

$$d_m = m\frac{\lambda}{2} \quad (\pm m = 0, 1, 2, 3, \cdots)$$

$$\therefore \lambda_1 = \frac{2d_1}{m}, \quad \lambda_2 = \frac{2d_2}{m}$$

$$\Delta\lambda = |\lambda_1 - \lambda_2| = \frac{2}{m}|d_1 - d_2| = \frac{2}{m}\Delta d. = \frac{2}{m}\delta_{1/2} \sim \frac{\lambda}{d}\delta_{1/2}$$

Here $\Delta d = \delta_{1/2}$ in (ii).

$$\therefore \Delta\lambda = \lambda\frac{\delta_{1/2}}{d} = \lambda\left(\frac{0.033}{2}\right) \text{ from (iii)}$$

4.

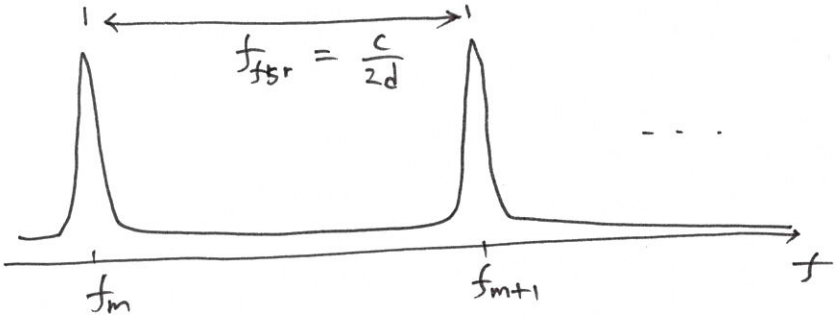
From #3,

The maximum transmission condition is

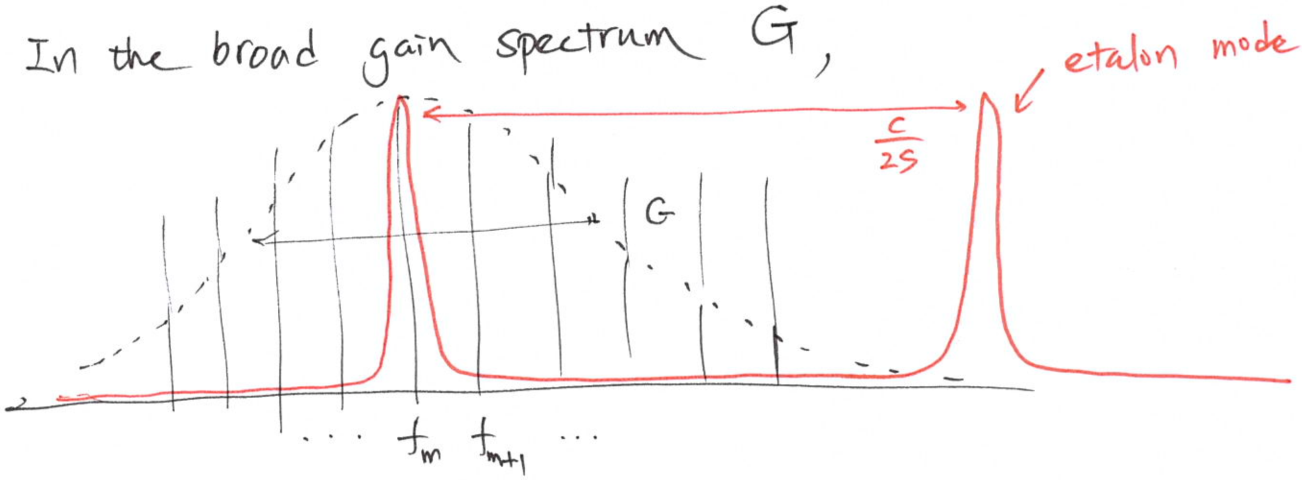
$$2kd = 2m\pi \rightarrow \frac{4\pi}{\lambda_m} d = 2m\pi$$

$$\lambda f = c, \rightarrow 4\pi \left(\frac{f_m}{c}\right) d = 2m\pi$$

$$\therefore f_m = m\left(\frac{c}{2d}\right)$$



In the broad gain spectrum G,



the etalon mode is chosen to pick one mode of the laser cavity, f_m.

$$\rightarrow \underline{\frac{c}{2S} > G} \rightarrow \boxed{S < \frac{c}{2G}}$$