• HW#8 6/8/2020

Sol. $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

(i) $AA^T = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$; $AA^T = U[\Sigma\Sigma^T]U^T$.

$|AA^T - \lambda I| = 0$, $\begin{vmatrix} 1-\lambda & -2 & -2 \\ -2 & 4-\lambda & 4 \\ -2 & 4 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)^2 + 16 + 16 - 4(4-\lambda) - 16(1-\lambda) - 4(4-\lambda)$

$= (4-\lambda)(16 - 8\lambda + \lambda^2) + 32 - 16 + 4\lambda - 16 + 16\lambda - 16 + 4\lambda$

$= 16 - 8\lambda + \lambda^2 - 16\lambda + 8\lambda^2 - \lambda^3 + 8\lambda + 16\lambda - 16$

$= 9\lambda^2 - \lambda^3 = -\lambda^2(\lambda - 9)$ $\therefore \lambda = 0, 9$.

- For $\lambda = 0$,

$[AA^T - \lambda I][x] = 0$, $\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ +2 \end{bmatrix}$

N. $1/\sqrt{18}$ (= $1/3\sqrt{2}$), $1/3$, $1/3$, $1/3$

- For $\lambda = 9$,

$\begin{bmatrix} -8 & -2 & -2 \\ -2 & -5 & 4 \\ -2 & 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

$\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \leftarrow \frac{1}{\sqrt{9}} = \frac{1}{3}$

· All eigen column vectors must be independent or orthogonal: $\sum_j^3 x_j^T x_j = 0 = \sum_j^3 x_j x_j^T$.

· All eight combinations satisfy this relation of orthogonality?

→ So, you'd can choose one of them, but satisfying $UU^T = I$.

ex) $U = \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}\frac{1}{3}$ → $UU^T \neq I$.

ex) $U = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\frac{1}{3}$ → $UU^T = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$. ✓

ex) $U = \begin{bmatrix} 1 & -2 & 2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix}\frac{1}{3}$ → $UU^T \neq I$.

(ii) $A^TA = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = [9]$.

$|A^TA - \lambda I| = 0$, $|9 - \lambda| = 0$ → $\lambda = 9$.

- For $\lambda = 9$, $[A^TA - \lambda I][x] = 0$, $[9 - 9][x] = 0$ $\therefore [x] = [1] = [x]^T$.

Thus, $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}[1]$. [9,0,0] -> [3,0,0]

$\text{or} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}[1]$.